Differential Equations Formulas
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First Order Ordinary Differential Equations

- **Existence-Uniqueness Theorem (Picard-Lindelöf):** For $y' = f(x, y), y(x_0) = y_0$: If f and $\partial f/\partial y$ are continuous in a rectangle around (x_0, y_0), a unique solution exists in some interval around x_0.
- **Separable Equations:** $\frac{dy}{dx} = g(x)h(y) \implies \int \frac{dy}{h(y)} = \int g(x)dx + C$
- **Linear Equations:** $y' + p(x)y = q(x)$. Integrating factor $\mu(x) = e^{\int p(x)dx}$. Solution: $y = \frac{1}{\mu(x)}\left(\int \mu(x)q(x)dx + C\right)$
- **Bernoulli Equations:** $y' + p(x)y = q(x)y^n$. Substitution: $v = y^{1-n}$. Transforms to linear: $v' + (1-n)p(x)v = (1-n)q(x)$
- **Homogeneous Equations:** $y' = F(y/x)$. Substitution: $v = y/x \implies y' = v'x + v$. Transforms to separable: $v'x + v = F(v) \implies \frac{dv}{F(v)-v} = \frac{dx}{x}$
- **Ricatti Equations:** $y' = P(x)y^2 + Q(x)y + R(x)$. Needs a particular solution y_1. Substitution $y = y_1 + u$ leads to a Bernoulli eq. for u.
- **Clairaut Equations:** $y = xy' + f(y')$. General solution: $y = cx + f(c)$ (family of lines). Singular solution: eliminate $p = y'$ from $y = xp + f(p)$ and $x + f'(p) = 0$.
- **Exact Equations:** $M(x, y)dx + N(x, y)dy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Solution $\Psi(x, y) = C$ where $\frac{\partial \Psi}{\partial x} = M$, $\frac{\partial \Psi}{\partial y} = N$.
- **Integrating Factors (μ):** $\mu M dx + \mu N dy = 0$ is exact. If $\frac{M_y - N_x}{N} = P(x)$, then $\mu(x) = e^{\int P(x)dx}$. If $\frac{N_x - M_y}{M} = Q(y)$, then $\mu(y) = e^{\int Q(y)dy}$.
- **Substitutions:** Various substitutions like $v = ax + by + c$ can simplify certain forms.
- **Applications:** Modeling population dynamics (logistic eq.), mixing problems, Newton's law of cooling, RC/RL circuits, orthogonal trajectories.

Higher Order Linear ODEs

- **Reduction of Order:** Given solution y_1 for $y'' + p(x)y' + q(x)y = 0$, find second solution $y_2 = y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1(x)^2}dx$.
- **Homogeneous Constant Coeff. Eq.:** $ay'' + by' + cy = 0$. Characteristic eq: $ar^2 + br + c = 0$. Roots r_1, r_2.
  - Distinct Real $r_1 \neq r_2$: $y_h = c_1e^{r_1x} + c_2e^{r_2x}$
  - Repeated Real $r_1 = r_2 = r$: $y_h = (c_1 + c_2x)e^{rx}$
  - Complex $r = \alpha \pm i\beta$: $y_h = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$
- **Undetermined Coefficients Method:** For $ay'' + by' + cy = g(x)$ where $g(x)$ is exp, poly, sin/cos or products/sums. Guess y_p based on $g(x)$'s form, modify if guess is part of y_h.
- **Variation of Parameters Method:** For $y'' + p(x)y' + q(x)y = g(x)$. Given $y_h = c_1y_1 + c_2y_2$. $y_p = u_1y_1 + u_2y_2$ where $u_1' = -\frac{y_2g}{W}$, $u_2' = \frac{y_1g}{W}$. Wronskian $W = y_1y_2' - y_1'y_2$.
- **Cauchy-Euler Equations:** Substitution: $x = e^t \implies t = \ln x$ and the following formulas

$$x\frac{dy}{dx} = \frac{dy}{dt}$$
$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$
$$x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$$

transforms the main equation into constant coefficient differential equation.

Using Laplace Transform for the Solution of Differential Equations

- **Laplace Transform Formulas:** Use Laplace Transform formulas given here. Click to see.

Series Solutions of ODEs

- **Classifying Points ($y'' + P(x)y' + Q(x)y = 0$):** Point x_0 is:
  Ordinary: if $P(x)$ and $Q(x)$ are analytic at x_0
  Singular: if not ordinary.
  Regular Singular: if x_0 is singular AND $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic at x_0.
  Irregular Singular: if x_0 is singular but not regular.
- **Power Series Solutions (Ordinary Point x_0):** Assume $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$. Substitute into ODE, find recurrence relation for a_n.
- **Power Series Solutions (Regular-Singular Point x_0):** Method of Frobenius: Assume $y = (x - x_0)^r \sum_{n=0}^{\infty} a_n(x - x_0)^n$. Find indicial equation for r. Find recurrence relation. Nature of solutions depends on roots r_1, r_2.

Systems of Linear ODEs

- **Solution Using Elimination:** For system $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$. Use differential operators (e.g., $D = d/dt$). Write system as polynomial equations in D acting on variables. Eliminate variables algebraically to get a single higher-order ODE for one variable. Solve it, then back-substitute.
- **Solution using Eigenvalues/vectors (Homogeneous):** For $\mathbf{x}' = A\mathbf{x}$. Find eigenvalues λ_i and eigenvectors $\mathbf{v}_i$ of matrix A. General solution is superposition: $\mathbf{x}(t) = c_1e^{\lambda_1 t}\mathbf{v}_1 + c_2e^{\lambda_2 t}\mathbf{v}_2 + \ldots$. Handle complex and repeated eigenvalues appropriately.

Fourier Analysis

- **Odd/Even/Periodic Functions:**
  Odd: $f(-x) = -f(x)$. Even: $f(-x) = f(x)$. Periodic (period P): $f(x + P) = f(x)$.
- **Trigonometric Series (Period $2L$):** $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$
  $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$
  $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \quad (n \geq 1)$
  $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx \quad (n \geq 1)$
- **Fourier Cosine Series (Even Ext., $0 \leq x \leq L$):** $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$. $b_n = 0$.
  $a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \quad (n \geq 0)$
- **Fourier Sine Series (Odd Ext., $0 \leq x \leq L$):** $f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$. $a_n = 0$.
  $b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx \quad (n \geq 1)$

Partial Differential Equations

- **Separation of Variables:** Assume solution form like $u(x, t) = X(x)T(t)$. Substitute into PDE. Separate variables to obtain ODEs for X and T. Solve ODEs using boundary conditions. Combine solutions via superposition (infinite series) to satisfy initial conditions.
- **1D Heat Equation:** $\alpha^2 u_{xx} = u_t$. Typical BCs: $u(0, t) = 0, u(L, t) = 0$. IC: $u(x, 0) = f(x)$. Solution often involves sine series: $u(x, t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi\alpha/L)^2 t} \sin\left(\frac{n\pi x}{L}\right)$.
- **1D Wave Equation:** $a^2 u_{xx} = u_{tt}$. Typical BCs: $u(0, t) = 0, u(L, t) = 0$. ICs: $u(x, 0) = f(x), u_t(x, 0) = g(x)$. Solution involves sine series with time-dependent coefficients: $u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi at}{L}\right) + B_n \sin\left(\frac{n\pi at}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$.
- **2D Laplace Equation (Rectangular):** $u_{xx} + u_{yy} = 0$. Requires BCs on all 4 sides. Solution via separation of variables often leads to terms like $(A_n \cosh(\frac{n\pi y}{L}) + B_n \sinh(\frac{n\pi y}{L}))\sin(\frac{n\pi x}{L})$ or similar, depending on which BCs are zero.